Equations Reducible to Linear Form:

I. Bernoulli's Eqn.

An eqn. of the form

$$\frac{dy}{dx} + Py = Qy^n \longrightarrow (1)$$

where P and Q are functions of x or constant, is called Bernoulli's eqn. Eqn. (1) can be reduced to Linear form as follows:

Dividing both sides by y^n, we get-

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \longrightarrow (2)$$

Now, we put-

$$\boxed{y^{-n+1} = v}$$

so that

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

Putting these values in (2), we get-

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

or, $$\frac{dv}{dx} + P(1-n)v = (1-n)Q$$

or, $$\frac{dv}{dx} + P_1 V = Q_1 \longrightarrow (3)$$

Where $P_1 = P(1-n)$, $Q_1 = (1-n)Q$.

Also, P_1 and Q_1 are functions of x.

Eqn. (3) is linear in v and x and can be solved as earlier.

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Eqns. reducible to Linear form:

II. Eqn. of the form

$$f'(y)\frac{dy}{dx} + P\,f(y) = Q \rightarrow (1)$$

where P and Q are functions of x or constants.

we put

$$\boxed{f(y) = v}$$

so that

$$f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ Putting in (1), we get

$$\frac{dv}{dx} + Pv = Q$$

which is a linear eqn. in v and x and can be solved as earlier. //

Ex. 1. Solve: $\frac{dy}{dx} + xy = xy^2 \longrightarrow (1)$

Soln. Diviring by y^2, we get

$$y^{-2}\frac{dy}{dx} + xy^{-1} = x \longrightarrow (2)$$

We put

$$y^{-1} = v \longrightarrow (i)$$

$$\Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dv}{dx} \rightarrow (ii)$$

$\therefore (2) \Rightarrow \; -\frac{dv}{dx} + xv = x$, by (i) & (ii)

$$\Rightarrow \frac{dv}{dx} - xv = -x \longrightarrow (3)$$

which is linear in v and x.

$$\therefore \text{I.F.} = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}$$

$\therefore$ General Solution is

$$v.(I.F.) = \int Q(I.F.)\,dx + c$$

$$\Rightarrow v.\,e^{-\frac{1}{2}x^2} = \int (-x).\,e^{-\frac{1}{2}x^2}\,dx + c$$

$$= \int e^t\,dt + c$$

let $-\frac{1}{2}x^2 = t$

$\Rightarrow -x\,dx = dt$

$$\Rightarrow y^{-1}\,e^{-\frac{1}{2}x^2} = e^t + c$$

$$= e^{-\frac{1}{2}x^2} + c$$

$$\Rightarrow y^{-1} = 1 + c\,e^{\frac{1}{2}x^2} \leftarrow \text{Ans.}$$

Q.2. Solve $\frac{dy}{dx} = x^3y^3 - xy$

Solⁿ Given eqn can be written as

$$\frac{dy}{dx} + xy = x^3y^3 \longrightarrow (1)$$

$$\Rightarrow y^{-3}\frac{dy}{dx} + xy^{-2} = x^3 \longrightarrow (2)\text{, dividing by } y^3.$$

Let $y^{-2} = v \longrightarrow (i)$

$$\Rightarrow (-2)y^{-3}\frac{dy}{dx} = \frac{dv}{dx}\text{, diff. w.r.t. '}x\text{'.}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx} \longrightarrow (ii)$$

$$\therefore (2) \Rightarrow -\frac{1}{2}\frac{dv}{dx} + xv = x^3\text{, by (i) \& (ii)}$$

$$\Rightarrow \frac{dv}{dx} - 2x.v = -2x^3$$

which is linear in v and x.

$$\therefore I.F. = e^{\int(-2x)\,dx} = e^{-x^2}$$

$\therefore$ G. Solⁿ is

i.e. $v \cdot (I.F.) = \int Q(I.F.)\,dx + c$

$\Rightarrow v \cdot e^{-x^2} = \int (-2x^3)(e^{-x^2})\,dx + c$

$= \int x^2(-2x)\,e^{-x^2}\,dx + c$

$= \int (-t)\,e^t\,dt + c$

Let $t = -x^2$

$\therefore dt = -2x\,dx$

$= -te^t - \int\left[\frac{d(-t)}{dt}\int e^t dt\right]dt + c$

$= -te^t + \int e^t\,dt + c$

$= -te^t + e^t + c$

Integration by parts

$\int uv = u\int v - \int[u'\int v\,dx]dx$

$\Rightarrow v \cdot e^{-x^2} = x^2 e^{-x^2} + e^{-x^2} + c$

$\Rightarrow v = x^2 + 1 + c\,e^{x^2}$

$\Rightarrow \frac{1}{y^2} = 1 + x^2 + c\,e^{x^2}$ ← Ans.

H.W. Solve

Ans:

① $\frac{dy}{dx}(x^2y^3 + xy) = 1$ $\left[\frac{1}{x} = (2 - y^2) - c\,e^{-\frac{1}{2}y^2}\right]$

Hint: treat x as dependant variable i.e. convert to the form $\frac{dx}{dy} + Px = Q$

② $2\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$ $\left[\frac{1}{y} = -c\,x^{-\frac{1}{2}} + x^{-1}\right]$

(3) $\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$, $\left[-\frac{1}{y}\sec^2 x = c + \frac{1}{3}\tan^3 x\right]$

(4) $(x - y^2)\,dx + 2xy\,dy = 0$ $\left[\frac{y^2}{x} = c - \log x\right]$