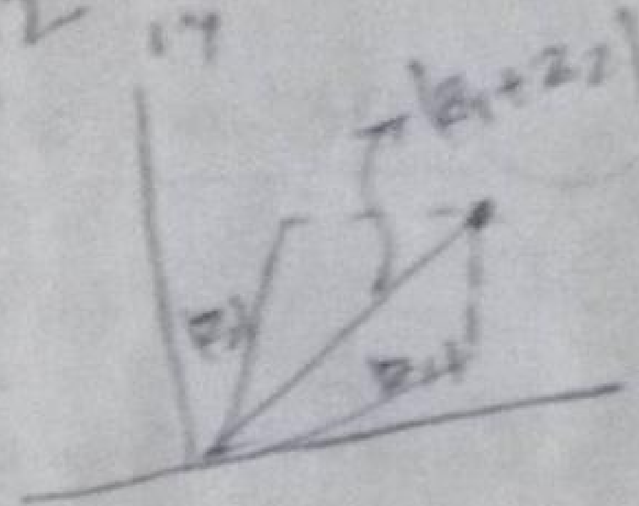
The equation ② represent the ...

Hence $|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$$= (|z_1| + |z_2|)^2$$



$$\therefore \quad |z_1+z_2| \le |z_1| + |z_2|$$

proved

De-Moiver's Theorem :-

If $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

& $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $z_1 z_2 = \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\}$

$$= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$

$$= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

The generalization of this eqn leads -

$$z_1 z_2 \cdots z_n = \{r_1 r_2 r_3 \cdots r_n\}\{\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\}$$

$\Rightarrow$ If $z_1 = z_2 = \cdots = z_n = z$, then

$$z^n = \{\cancel{r(\cos\theta + i\sin\theta)}\}^n = \{r(\cos n\theta + i\sin n\theta)\}^1$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This is known as De-Moivre's Theorem. 3

Roots of complex numbers

If a number ω is called an nth root of a complex number z if $w^n = z$

$\Rightarrow \quad \omega = z^{1/n}$

$\therefore \quad z^{1/n} = \{ r(\cos\theta + i\sin\theta) \}^{1/n}$

$= r^{1/n} \{ \cos\left(\theta_p + \frac{2k\pi}{n}\right) + i\sin\left(\theta_p + \frac{2k\pi}{n}\right)$

where $k = 0, 1, 2, \ldots, (n-1)$

where $0 \le \theta_p < 2\pi$.

Problem: find the square root of i

Soln: Here $z = i$

$\therefore (i)^{1/2} = r^{1/2}\left[\cos\left(\frac{\theta_p + 2\pi k}{2}\right) + i\sin\left(\frac{\theta_p + 2\pi k}{2}\right)\right]$

The principal $\arg z = \theta = \theta_p = \pi/2$, because $x = 0$, $y = 1$, $r = 1$

For $k = 0$, we obtain

$(i)^{1/2}_{k=0} = \cos\pi/4 + i\sin\pi/4$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$

For $k = 1$, $(i)^{1/2}_{k=1} = \cos(\pi/4 + \pi) + i\sin(\pi/4 + \pi)$

$= \cos\pi/4 \cos\pi + i\sin\pi/4 \cdot \cos\pi$

$\Delta x \to 0$

$\Delta y \to 0$

S-V/L-3

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Roots of complex number :- If a number ω is called an nth of a complex no. z if $\omega^n = z$

$$\Rightarrow \omega = z^{\frac{1}{n}}$$

$$z^{\frac{1}{n}} = \left\{ r(\cos\theta + i\sin\theta) \right\}^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} \left\{ \cos\left(\frac{\theta_p + 2k\pi}{n}\right) + i\sin\left(\frac{\theta_p + 2k\pi}{n}\right) \right.$$

where $k = 0, 1, 2, \ldots (n-1)$; where $0 \le \theta_p \le 2\pi$

Problem :- Find the square root of i

Soln $z = i$.

$$(i)^{\frac{1}{2}} = r^{\frac{1}{2}} \left\{ \cos\left(\frac{\theta + 2\pi k}{\cancel{n}\,2}\right) + i\sin\left(\frac{\theta_p + 2\pi k}{2}\right) \right\}$$

because $x = 0$, $y = 1$, $r = 1$, $\theta_p = \pi/2$

(a) For $k = 0$, we obtain,

$$(i)^{\frac{1}{2}}_{k=0} = \cos\left(\frac{\pi/2 + 0}{2}\right) + i\sin\left(\frac{\pi/2 + 0}{2}\right)$$

$$= \frac{1}{\sqrt{2}} + i/\sqrt{2}$$

$$= \frac{1}{\sqrt{2}}(1 + i)$$

(b) $k = 1$, we obtain

$$(i)^{\frac{1}{2}}_{k=1} = \cos\left(\frac{\pi/2 + 2\pi}{2}\right) + i\sin\left(\frac{\pi/2 + 2\pi}{2}\right)$$

$$= \cos\pi/4 \cos\pi + i\sin\pi/4 \cancel{\sin\pi} \cos\pi$$

$$= -\frac{1}{\sqrt{2}} - i/\sqrt{2} = -\frac{1}{\sqrt{2}}(1 + i)$$

$$\Rightarrow (i)^{1/2}_{k=1} = -\frac{1}{\sqrt{2}}(1 + i)$$

Hence roots $(i)^{1/2} = \pm \frac{1}{\sqrt{2}}(1 + i)$ //

$\Delta x \to 0$

$\Delta y \to 0$

The equation (2) represent the

Problem

$= \frac{-1}{\sqrt{2}}(1+i)$.

(b) Euler's Formula :-

For real θ, we may write the infinite series expansion of holds when $x = i\theta$

$e^x = 1 + x + \frac{x^2}{2!} + x^3/3! + \cdots$

$e^{i\theta} = \cos\theta + i\sin\theta$ which is called Euler's Formula.

where $e = 2.71828$ //

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Again, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$

Putting $x = z$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, when $z = i\theta$

$\Rightarrow e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$

$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$

$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$

$= \cos\theta + i\sin\theta$

Again, $z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ //

=×=

. . sufficient condition for $f(z)$ to be

~~Dot and cross product~~ 3

Prob If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ Prove ~~that~~ that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$ by using ~~the~~ Euler's formula

Solⁿ

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} \quad // = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \quad //$$

Again we know that from Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{so that } z_1 = r_1e^{i\theta_1}, \quad z_2 = r_2e^{i\theta_2}$$

$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i\theta_1}/e^{i\theta_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

//

Problem Prove the identities

(a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) $\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$; if $\theta \neq 0, \pm\pi, \pm 2\pi$ -

Solⁿ we use the ~~binom~~ binomial formula →

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$

$\Delta x \to 0$

$\Delta y \to 0$

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Solution $= \frac{-1}{5}$ ...

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ known as ${}^nC_r$ the binomial co-efficient

For $n! = 1, 2, 3, \ldots n$ and $0! = 1$

when $n = 5$,

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + \binom{5}{1}(\cos^4\theta)(i\sin\theta) + \binom{5}{2}\cos^3\theta(i\sin\theta)^2 + \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

$\therefore$ Hence $\underline{\cos 5\theta} = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 20\cos^3\theta + 5\cos\theta$ //

$= 16\cos^5\theta - 20\cos^3\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

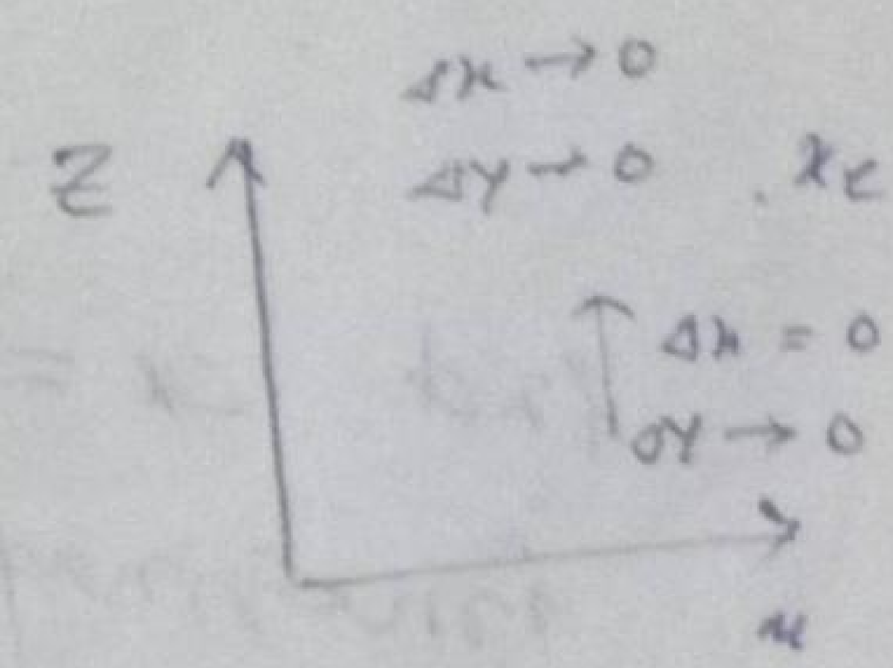
$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$ //

Next

Necessary and sufficient condition for $f(z)$ to be analytical :-

If $w = f(z) = u(x,y) + iv(x,y)$ is an analytical in a domain, then that domain should

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\}$$ These equations are called Cauchy's - Riemann Conditions.



Proof.

Let $z = x + iy$

$f(z) = u(x,y) + iv(x,y)$

where u & v are real functions of real variable x and y. Let the increments Δx & Δy of the variable x & y then

$$\Delta z = \Delta x + i\Delta y$$

$\therefore$ change in function $f(z)$ is $\Delta f(z) = \Delta w$.

$$\Delta w = f(z+\Delta z) - f(z)$$

$$\therefore \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u(x,y) + i\Delta v(x,y)}{\Delta x + i\Delta y}$$

The equation (2) represent ...

where $\frac{n}{r} = \frac{n!}{\quad}$

We have two different approaches as shown, the figure.

First $\Delta y = 0$, we get $\Delta x \to 0$ i.e Δz is wholly real

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u(x,y) + i\Delta v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2nd $\Delta x = 0$ we get $\Delta y \to 0$ i.e. Δz is wholly imaginary

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u(x,y) + i\Delta v(x,y)}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since the $f(z)$ is ~~analytical~~ analytic at then its derivative must exist and identical.

So, that

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \; //$$

$\frac{\partial v}{\partial x} =$. Cauchy - Riemann Condition

$\frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial y}$

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$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy Riemann Condition for analytic