Determination of polynomial when some functional values of the function are given:→

Ex.
What is the lowest degree polynomial which takes the following values

x	0	1	2	3	4	5
$f(x)$	0	3	8	15	24	35

Solⁿ. We have

$$f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + {}^nC_n \Delta^n f(a) \rightarrow (1)$$

Putting $a=0$, $h=1$, $n=x$ in (1), we get

$$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(0) + \cdots + {}^xC_x \Delta^n f(0)$$

$$= f(0) + x^{(1)} \Delta f(0) + \frac{x^{(2)}}{2!} \Delta^2 f(0) + \cdots \rightarrow (2)$$

$$\left[\because x^{(n)} = \frac{x!}{(x-n)!}\right]$$

$${}^xC_2 = \frac{x!}{2!(x-2)!} = \frac{x^{(2)}}{2!}$$

We now construct the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0			
		3		
1	3		2	
		5		0
2	8		2	
		7		0
3	15		2	
		9		0
4	24		2	
		11		
5	35			

Putting these values of $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$ in (2), we get –

$$f(x) = 0 + x^{(1)} \cdot 3 + x^{(2)} \frac{1}{2!} \cdot 2 + 0 = 3x^{(1)} + x^{(2)} = 3x + x(x-1)$$

$$\Rightarrow f(x) = x^2 + 2x \leftarrow \text{Ans.}$$

—x—

Ex. Find the polynomial of lowest degree which Satisfies the following set of numbers

0, 7, 26, 63, 124, 215, 342, 511.

Solⁿ. First we construct the following diff. table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		0
		61		6	
4	124		30		0
		91		6	
5	215		36		0
		127		6	
6	342		42		
		169			
7	511				

Now, we know,

$$f(a+nh) = f(a) + {}^{n}C_1 \Delta f(a) + {}^{n}C_2 \Delta^2 f(a) + \cdots + {}^{n}C_n \Delta^n f(a) \longrightarrow (1)$$

Putting $a=0$, $h=1$, $n=x$ in (1), we get

$$f(x) = f(0) + {}^{x}C_1 \Delta f(0) + {}^{x}C_2 \Delta^2 f(0) + {}^{x}C_3 \Delta^3 f(0) + \cdots \longrightarrow (2)$$

$$\Rightarrow f(x) = 0 + {}^{x}C_1 \cdot 7 + {}^{x}C_2 \cdot 12 + {}^{x}C_3 \cdot 6 + 0 \cdots$$

$$= 0 + 7x^{(1)} + 12 \cdot \frac{x^{(2)}}{2!} + 6 \cdot \frac{x^{(3)}}{3!} + 0 \cdots$$

$$= 7(x) + 6x(x-1) + x(x-1)(x-2)$$

$$= 7x + 6x^2 - 6x + x^3 - 3x^2 + 2x$$

$$\Rightarrow f(x) = x^3 + 3x^2 + 3x \longleftarrow \text{Ans.}$$

Ex. Find the function whose first difference is e^x.

Sol^n: We know that

$$\Delta e^x = e^{x+h} - e^x \quad , h \text{ being the interval of differencing}$$

$$= e^x . e^h - e^x$$

$$= e^x (e^h - 1)$$

$$\Rightarrow e^x = \frac{1}{e^h - 1} \Delta e^x$$

$$= \Delta \left[\frac{e^x}{e^h - 1} \right]$$

i.e. we get,

$$\Delta \left[\frac{e^x}{e^h - 1} \right] = e^x .$$

∴ The required function

$$f(x) = \frac{e^x}{e^h - 1} \longleftarrow \text{Ans.}$$

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