The vibration of the air inside an organ pipe can be shown by bringing a light paper pan having some fine dry sand on it close to its end. It will be noticed that the sand particles dance about while the pipe is sounding. Thus we see that in every case the body producing sound is in a state of vibration. Most sounding bodies, unless their motion be very great, execute a particular type of periodic motion *i.e.*, a motion which repeats itself over and over again after regularly recurring intervals, called **simple harmonic motion** or their vibration consists of a combination of two or more simple harmonic motions and hence the importance of this type of motion in the study of sound.

1.2. Simple Harmonic Motion. Let a particle P move with uniform speed, v, round a circle (Fig. 1-2) and let N be the foot of the perpendicular drawn from P on the diameter BB'. Now consider the movement of N as P moves once round the circle in the anti-clokwise direction starting from A. When P is at A, N is at the centre O; when P reaches B, N moving along OB coincides with B. As P moves along the circumference from B to B', N travels from B to B' along the diameter and as P finally returns to A completing one revolution, N returns to O having traversed the diameter BB' twice. This oscillatory motion of N about the centre O is '**simple harmonic motion**'. **It may be regarded as the projection of uniform circular motion upon any diameter of the circle which is referred to as the circle of reference and the point P is called the reference point.**

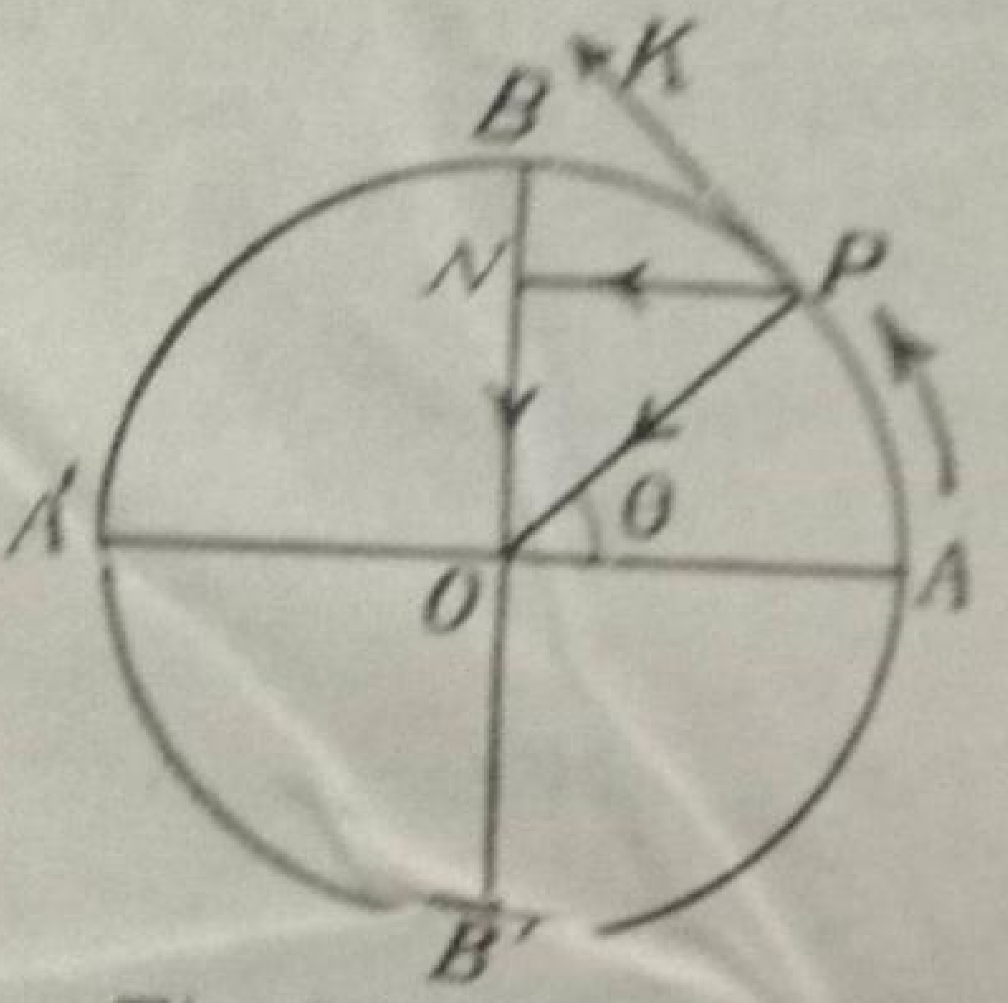

Fig. 1 2. Simple Harmonic Motion.

Let a be the radius of the circle and θ the angle at any instant which the rotatory vector OP makes with OA, the line joining the initial positions of P and N. The angular velocity ω of P is v/a and the *displacement* of N from O is given by

$$y=a \sin \theta.$$

If t seconds be the time in which OP describes the angle θ, then $\theta=\omega t$ and we may write

$$y=a \sin \omega t. \qquad (1)$$

The radius a of the circle of reference or the maximum displacement of N on either side of O is called its **amplitude**.

The velocity of N is clearly given by the component of the velocity of P parallel to the diameter BB'. Now the velocity of P is v along PK tangential to the circle at P. Its component parallel to the diameter BB' *ie*. perpendicular to PN is $v \sin \angle KPN = v \cos \theta$, since θ and $\angle KPN$ are complementary.

Therefore the velocity of $N = v \cos \theta$

$$= v \cos \omega t$$

$$= v\sqrt{1-\sin^2 \omega t}.$$

Now $\sin \omega t = \frac{y}{a}$ from equation (1)

and $v = a\omega,$

$\therefore$ **velocity of N** $= a\omega\sqrt{1-\frac{y^2}{a^2}}$

$$= a\omega\sqrt{\frac{a^2-y^2}{a^2}}$$

$$= \omega\sqrt{a^2-y^2} \qquad (2)$$

Thus the velocity of N varies along the path depending upon its displacement *i.e.*, the distance from the mean position O, being a maximum when y is minimum and minimum when y is maximum.

It can easily be seen that the maximum value of velocity is v or $a\omega$ and occurs when the particle passes its mean position in either direction and the velocity is zero when it has attained its maximum displacement on either side.

The acceleration of N is the component along NO of the centripetal acceleration of P. Since P is moving in a circle the acceleration of P is $v^2/a = a\omega^2$ directed towards the centre, therefore

the acceleration of $N = -a\omega^2 \sin\theta$

$$= -\omega^2(a \sin\theta)$$

$$= -\omega^2 y, \qquad (3)$$

negative sign indicates that it acts opposite to the direction in which y increases.

Obviously, this acceleration has its maximum value $a\omega^2$, the same as that of P and this occurs at the extreme positions B and B' when $y=a$; and its minimum value is zero and occurs at C when $y=0$.

From equation (3) Simple Harmonic motion may be defined as follows :

A body is said to move in 'Simple Harmonic Motion' (written for brevity as SHM) if the acceleration ($\omega^2 y$) is always directed towards a fixed point in the centre and is proportional to its distance from it.

It should be noted that the path of the particle may not necessarily be straight but may as well be curved.

The 'periodic time' of the motion is equal to the time occupied by P in moving once round the circle. Since the angular velocity is ω, the periodic time $T = 2\pi/\omega$ and the frequency *i.e.*, the number of vibrations per sec is given by

$$\boxed{n = \frac{1}{T} = \frac{\omega}{2\pi}} \qquad (4)$$

Equation (1) giving the displacement of the particle may therefore be written as

$$y=a \sin 2\pi t/T=a \sin 2\pi nt. \tag{5}$$

Also from equation (3) we learn that ω^2 is the acceleration when the particle is at a unit distance from the centre. We have, therefore,

$$T=2\pi/\sqrt{\text{acceleration per unit displacement}}$$

$$=2\pi/\sqrt{\frac{\text{acceleration}}{\text{displacement}}}=2\pi/\sqrt{\frac{\text{displacement}}{\text{acceleration}}} \tag{6}$$

The time period is independent of the amplitude, *i.e.*, the vibrations are *isochronous**, whatever the nature of the path, straight or curved.

1·3. Phase and Epoch. We have seen above that the displacement of N is $y=a \sin \omega t$, on the assumption that we start counting time when P crosses the x-axis. If instead of measuring time from this particular instant, it is measured from an instant when P is at P_0 where $\angle AOP_0=\phi$, the displacement ON is given by $y=a \sin (\omega t-\phi)$ since $\angle AOP=(\omega t-\phi)$. The angle $(\omega t-\phi)$ is called the **phase of the vibration.** The value of the phase when $t=0$, *viz.* $-\phi$ is termed the **'epoch'** or *the initial phase*.

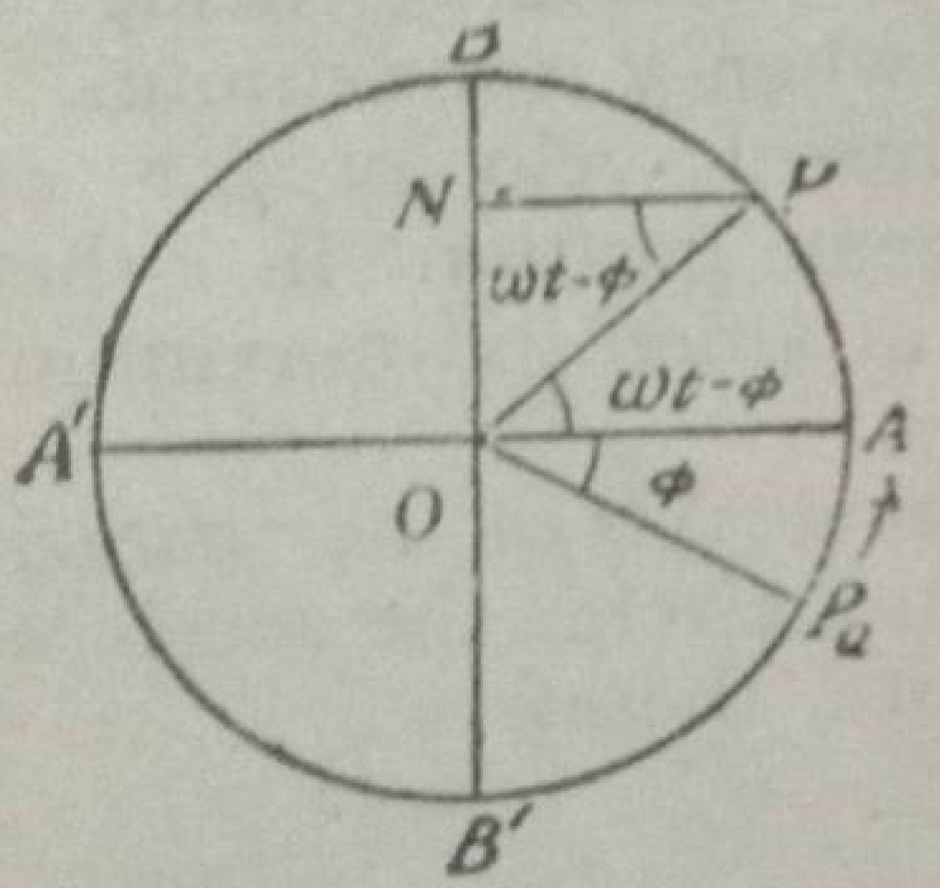

Fig. 1·3. Explaining Phase and Epoch.

If, however, we start counting time when P has already crossed the x-axis and its initial position is on the other side of A, then calling this initial angle ϕ as before, the displacement after time t is given by $y=a \sin (\omega t+\phi)$, so $(\omega t+\phi)$ is the phase of the vibration.

By the term *'Phase'* of a harmonically vibrating point at any particular instant we mean its state as regards its position and its direction of motion at that instant. It is the value of the angle AOP and denotes what stage of the cycle the particle has reached at that instant. If it is zero, the particle is at the position of zero displacement and moving towards B. If it is $\pi/2$, the particle is at the position of the maximum positive displacement and so on. It is convenient sometimes to express **'phase' of a harmonically vibrating point as a fraction of the time period which has elapsed since the point last passed through its mean position in the positive direction.**

The term **phase difference** between two SH Vibrations indicates how much the two vibrations are out of step with each other. Thus if two vibrating points pass simultaneously through their undisplaced positions in the same direction, that is, the phase difference at any instant is an integral multiple of 2π, they are at the instant of such passage said to be *in phase*. If, however, the direc-

*Vibration whose periods do not alter are called 'isochronous' but note that it is not necessary that isochronous vibrations may necessarily be harmonic.

tion of one of them is reversed, all else remaining the same, they would then be in *opposite* phase or differ in phase by 180°. If the phase difference is $\pi/2$, the motions are said to be in *quadrature*. This difference in phase can also be expressed in terms of the periodic time of one or the other vibration.

We conclude from the above discussion that the three main characteristics of SH Motion are its *period*, its *amplitude* and its *phase* embodied in the equation of its displacement

$$y = a \sin(\omega t \mp \phi) \quad (7)$$

$$= a \sin\left(\frac{2\pi t}{T} \mp \phi\right), \text{ since } \omega = \frac{2\pi}{T}$$

1.4. Differential Equation of SHM. Using the language of differential calculus, by differentiating equation (1) on page 2 with respect to time we have for the velocity of N,

$$\frac{dy}{dt} = a\omega \cos \omega t.$$

Differentiating again, we get the acceleration of N,

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t$$

$$= -\omega^2 y, \quad (8)$$

or

$$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0,} \quad (9)$$

where ω^2 is constant.

This is the "**differential equation**" of Simple Harmonic Motion.

It may be interesting for the student to acquaint himself with the mathematics of solving the differential equation of SHM. To do so we have to integrate equation (8), but as it cannot be integrated directly we proceed as follows :

Let $v = \frac{dy}{dt}$, so that $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$

$$= \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v\frac{dv}{dy}$$

Equation (8) therefore becomes

$$v\frac{dv}{dy} = -\omega^2 y,$$

or

$$v\,dv = \omega^2 y\,dy,$$

Integrating it we have

$$\int v\,dv = \int -\omega^2 y\,dy = -\omega^2 \int y\,dy,$$

or

$$\frac{v^2}{2} = -\frac{\omega^2}{2} y^2 + C,$$

where C is a constant of integration and has to be determined from the condition of the particle at the instant considered.

For extreme displacement,

$$y=a \quad \text{or} \quad -a \text{ and } v=0,$$

$$\therefore \qquad 0=-\frac{\omega^2a^2}{2}+C \text{ or } C=\frac{\omega^2a^2}{2}$$

whence,

$$\frac{v^2}{2}=\frac{\omega^2a^2}{2}-\frac{\omega^2y^2}{2},$$

or

$$v^2=\omega^2(a^2-y^2),$$

or

$$v=\omega\sqrt{a^2-y^2}.$$

This gives the velocity of the particle at any instant and is the same as equation (2) on page 3.

Writing $\frac{dy}{dt}$ for v, we have

$$\frac{dy}{dt}=\omega\sqrt{a^2-y^2},$$

or

$$\frac{dy}{\sqrt{a^2-y^2}}=\omega dt.$$

Integrating it, we have

$$\int\frac{dy}{\sqrt{a^2-y^2}}=\int\omega dt,$$

or

$$\sin^{-1}\frac{y}{a}=\omega t+\phi,$$

ϕ being a constant of integration determined by initial conditions.

It may be put as

$$y=a\sin(\omega t+\phi). \tag{10}$$

Expanding the equation, we get

$$y=a\cos\phi\sin\omega t+a\sin\phi\cos\omega t,$$

where $a\cos\phi$ and $a\sin\phi$ are just new constants. Let us represent them by A and B.

Clearly then our equation could be expressed by the formula

$$y=A\sin\omega t+B\cos\omega t, \tag{11}$$

and in special cases either A or B may be zero.

The period of vibration can be easily deduced. If we start counting time when the displacement is maximum, that is, $t=0$ when $y=a$, we have from equation (10),

$$\sin\phi=1 \quad \text{or} \quad \phi=\pi/2 \text{ and}$$

$$y=a\sin(\omega t+\pi/2)=a\cos\omega t.$$

Every time ωt increases by 2π, the displacement repeats itself since $\cos(\omega t+2\pi)=\cos\omega t$.

Writing $a\cos(\omega'+2\pi)=a\cos\omega(t+2\pi/\omega)$, we find that y repeats itself at intervals of $2\pi/\omega$. Hence the periodic time T is equal to $2\pi/\omega$.