§. To Express any Value of the function in Terms of Leading Term and the Leading Differences of Difference Table :-

To show that

$$\boxed{f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + {}^nC_n \Delta^n f(a)} \rightarrow (A)$$

Sol^n: We know that

$$\Delta f(a) = f(a+h) - f(a)$$

$$\therefore f(a+h) = f(a) + \Delta f(a) \longrightarrow (1)$$

Again,

$$\Delta f(a+h) = f(a+2h) - f(a+h)$$

$$\therefore f(a+2h) = f(a+h) + \Delta f(a+h)$$

$$= f(a) + \Delta f(a) + \Delta[f(a) + \Delta f(a)], \text{ by } (1)$$

$$= f(a) + 2\Delta f(a) + \Delta^2 f(a) \rightarrow (2)$$

Similarly,

$$\Delta f(a+2h) = f(a+3h) - f(a+2h)$$

$$\therefore f(a+3h) = f(a+2h) + \Delta f(a+2h)$$

$$= f(a) + 2\Delta f(a) + \Delta^2 f(a) + \Delta[f(a) + 2\Delta f(a) + \Delta^2 f(a)], \text{ by } (2)$$

$$= f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a) \longrightarrow (3)$$

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Thus, we see that the given result(*) is true for $n = 1, 2$ and 3.

Now, we assume that the given result

$$f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + {}^nC_n \Delta^n f(a) \rightarrow (*)$$

be true for any value n.

We shall show that this result is true for the next integer $n+1$.

Now, we get

$$\Delta f(a+nh) = f(a+\overline{n+1}\,h) - f(a+nh)$$

$$\Rightarrow f(a+\overline{n+1}\,h) = f(a+nh) + \Delta f(a+nh)$$

$$= \left[f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + {}^nC_n \Delta^n f(a)\right]$$

$$+ \Delta\left[f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + \cdots + {}^nC_n \Delta^n f(a)\right]$$

$$= f(a) + \left[{}^nC_1 \Delta f(a) + \Delta f(a)\right]$$

$$+ \left[{}^nC_2 \Delta^2 f(a) + {}^nC_1 \Delta^2 f(a)\right]$$

$$+ \cdots\cdots + {}^nC_n \Delta^{n+1} f(a).$$

$$= f(a) + {}^{n+1}C_1 \Delta f(a) + {}^{n+1}C_2 \Delta^2 f(a)$$

$$+ \cdots + \Delta^{n+1} f(a)$$

$$\left[\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}\right]$$

Hence the result is true for $n = n+1$;

Hence by mathematical induction, the given result is true for all integral value of n.

Hence proved.

Factorial Notation :

The product of factors in which the first factor is x and the successive factors decrease by a constant difference is known as a factorial function and is denoted by $x^{(n)}$, where n is a positive integer. These functions play an important role in the theory of finite differences.

$$x^{(n)} = x(x-1)(x-2)(x-3)\cdots(x-\overline{n-1}) \rightarrow (1)$$

[when the interval of differencing is unity]

$$x^{(n)} = x(x-h)(x-2h)(x-3h)\cdots(x-\overline{n-1}h) \longrightarrow (2)$$

[when the interval of differencing is h]

Again from (1),

$$x^{(n)} = \frac{x(x-1)(x-2)(x-3)\cdots(x-\overline{n-1})(x-n)!}{(x-n)!}, \quad x>n$$

$$\Rightarrow x^{(n)} = \frac{x!}{(x-n)!}$$

Q. To show that

$$\boxed{\Delta^{n} x^{(n)} = n!\,h},$$

and, $\boxed{\Delta^{n+1} x^{(n)} = 0}$

Proof :→ H.W.