Sol^n of lin. diff. eq^n of the form

$$\frac{dx}{dy} + Px = Q \longrightarrow (1)$$

where x is dependent variable, y is independent variable & P, Q are f^{ns} of y or constants.

In this case G.S. is

$$x.(I.F.) = \int Q.(I.F.)\,dy + c \longrightarrow (2)$$

where

$$I.F. = e^{\int P\,dy} \longrightarrow (3)$$

Ex.1. Solve

$$(x+y+1)\frac{dy}{dx} = 1$$

Sol^n Given eq^n is

$$(x+y+1)\frac{dy}{dx} = 1 \longrightarrow (1)$$

It can be written as

$$\frac{dx}{dy} = x+y+1$$

or, $\dfrac{dx}{dy} - x = 1+y$, [It is of the form $\dfrac{dx}{dy} + Px = Q$]

which is linear.

Here, $P = -1$
$Q = 1+y$.]

$$I.F. = e^{\int P\,dy}$$

$$= e^{\int(-1)\,dy} = e^{-y}.$$

Hence, the gen. sol^n of the eq^n (1) is

$$x.(I.F.) = \int Q(I.F.)\,dy + c$$

or, $x.\,e^{-y} = \int (1+y)\cdot e^{-y}\,dy + c$

or,

$$x \cdot e^{-y} = \int e^{-y}\,dy + \int y e^{-y}\,dy + c$$

$$= -e^{-y} + y\int e^{-y} - \int\left[\frac{d}{dy}(y)\cdot\int e^{-y}\,dy\right]dy + c$$

(Integrating by parts.)

$$= -e^{-y} - ye^{-y} + \int e^{-y}\,dy + c$$

$$= -e^{-y} - ye^{-y} - e^{-y} + c$$

$$\Rightarrow xe^{-y} = -ye^{-y} - 2e^{-y} + c$$

$$\Rightarrow x = -y - 2 + ce^{y}$$

$$\Rightarrow x + y + 2 = ce^{y} \longleftarrow \text{Ans.}$$

Ex. 2. Solve $(x+2y^3)\dfrac{dy}{dx} = y$

Sol^n^ Given eqn. can be written as

$$\frac{dx}{dy} = \frac{(x+2y^3)}{y}$$

or, $$= \frac{x}{y} + 2y^2$$

or, $$\frac{dx}{dy} - \frac{x}{y} = 2y^2 \longrightarrow (1)$$

or, $\dfrac{dx}{dy} + Px = Q$, where $P = -\dfrac{1}{y}$, $Q = 2y^2$

which is linear in y and x.

$$\therefore \; I.F. = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

$\therefore$ G.S. of (1) is

$$x\cdot(I.F) = \int Q\,(I.F.)\,dy + c$$

$$\Rightarrow x\cdot\left(\frac{1}{y}\right) = \int 2y^2\cdot\frac{1}{y}\,dy + c = 2\int y\,dy + c = y^2 + c$$

$$\Rightarrow x = y^3 + cy \longleftarrow \text{Ans.}$$

Ex. 3. Solve $(1+y^2)\,dx + (x - \tan^{-1} y)\,dy = 0$

Soln. The given eqn. can be written as

$$(1+y^2)\frac{dx}{dy} + x - \tan^{-1}y = 0$$

or, $$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1}y}{1+y^2} \rightarrow (1)$$

which is of the form $\frac{dx}{dy} + Px = Q \rightarrow (2)$

where $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$.

Its I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore$ G.S. is

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \cdot e^{\tan^{-1}y}\,dy + c$$

$$= \int t e^t\,dt + c$$

$$= te^t - \int\left(\frac{dt}{dt}\int e^t dt\right)dt + c$$

Let $\tan^{-1}y = t$
$\therefore \frac{1}{1+y^2}dy = dt$

$$= te^t - \int e^t dt + c$$

$$= te^t - e^t + c$$

$$= e^t(t-1) + c$$

$$\Rightarrow x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\Rightarrow x = (\tan^{-1}y - 1) + c \cdot e^{-\tan^{-1}y} \quad \leftarrow \text{Ans.}$$

H.W. Solve

Ans:

(1) $dx + x\,dy = e^{-y}\log y\,dy$ $\quad [\,xe^y = c + y\log y - y\,]$

(2) $(x+2y^3)\frac{dy}{dx} = y$ $\quad [\,x + 2y^3 + 6y^2 + 12y + 12 = ce^y\,]$

(3) $\frac{dy}{dx} + y\cos x = \sin x\cos x$ $\quad [\,y = \sin x - 1 + ce^{-\sin x}\,]$

(4) $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$, given $y(0)=0$ $\quad [\,3y(1+x^2) = 4x^3\,]$

[Hints for (4): In G.S., we put $x=0, y=0$ to get c]