Ex. Evaluate $\Delta^3 f(x)$, where $f(x) = (1-x)(1-2x)(1-3x)$, the interval of differences being unity.

Solⁿ. Given

$$f(x) = (1-x)(1-2x)(1-3x)$$
$$= (1-x)(1-5x+6x^2)$$
$$= 1-5x+6x^2-x+5x^2-6x^3$$
$$= -6x^3+11x^2-6x-1.$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$\therefore \Delta^n f(x) = \lfloor n \, h^n a_n$

where h is the interval of differences

i.e. $f(x)$ is a polynomial of degree 3.

$$\therefore \Delta^3 f(x) = (-6)\lfloor 3$$
$$= (-6).3.2.1$$
$$= -36 \leftarrow \text{Ans.}$$

Ex. Evaluate

$$\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right].$$

Solⁿ. Clearly $f(x) = (1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$ will be a polynomial of degree 10. Also the co-efficient of x^{10} will be $abcd$.

$\boxed{\Delta^n f(x) = \lfloor n \, h^n a_n}$

$$\therefore \Delta^{10} f(x) = \lfloor 10 . 1^{10} . abcd$$
$$= abcd \lfloor 10 . \leftarrow \text{Ans}$$

(here 1 is the interval of differences.

One or More Missing terms:

To find the missing terms of the values of the function $y = f(x)$. We have two methods:

Method. 1:

Let n values out of $(n+1)$ values of $y = f(x)$ are given for $(n+1)$ values of x, the values of x being equidistant. Let the unknown value be M. We then construct the difference table. Since only n values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $(n-1)$ in x. Now equating to zero the difference we get the value of M.

Method 2.

In case, we know the value of the diff. Then

$$\Delta^n f(x) = \sum_{i=0}^{n} (-1)^{n-i} \, {}^nC_i \, f(x+ih),$$

can be used to estimate the missing term.

Let $f(x)$ be a poly. of degree $(n-1)$ in x.

Then $\Delta^n f(x) = 0$ for all x.

$$\therefore \sum_{i=0}^{n} (-1)^{n-i} \, {}^nC_i \, f(x+ih) = 0$$

From this eqn., we can find the missing term.

Note: Generally we use Method 1.

Ex. Estimate the missing term in the following table:

x	0	1	2	3	4
$y=f(x)$	1	3	9	.....	81

Soln. We construct the following difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
			4		
1	3			y_3-19	
		6			
			y_3-15		$124-4y_3$
2	9				
		y_3-9		$105-3y_3$	
3	y_3		$90-2y_3$		
		$81-y_3$			
4	81				

We are given four values of $y=f(x)$, so 3rd differences are constant and consequently 4th difference is zero.

$$\therefore \Delta^4 y = 0$$

$$\Rightarrow 124-4y_3 = 0, \quad \text{from table}$$

$$\Rightarrow y_3 = 31$$, which is the required missing term.

Ex. Obtain the missing terms in the following table:

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	...	64	...	216	343	512

Solution: We are given six values, so fith differences are constant and consequently sixth differences are zero,

i.e. $\Delta^6 f(x) = 0 \quad \forall x$

or, $(E-1)^6 f(x) = 0$

or, $(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0$

$[\because 1+\Delta \equiv E]$

or, $f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0$

Putting $x=1$ and (2), we get

$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0 \rightarrow$ (i)

& $f(8) - 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0 \rightarrow$ (ii)

Putting the values of $f(8), f(7), f(6), f(4), f(2), f(1)$ in the eqn (i) & (ii), we get,

(i) $\Rightarrow 343 - 6\times216 + 15f(5) - 20\times64 + 15f(3) - 6\times8 + 1 = 0$

$\Rightarrow 343 - 1296 + 15f(5) - 1280 + 15f(3) - 48 + 1 = 0$

$\Rightarrow 15f(5) + 15f(3) = 2280$

$\Rightarrow f(5) + f(3) = 152 \longrightarrow$ (iii)

$343 + 1 = 344$

$1296 + 1280 + 48 = 2624$

$2624 - 344 = 2280$

Again, (ii) ⇒

$512 - 6\times343 + 15\times216 - 20f(5) + 15\times64 - 6f(3) + 8 = 0$

$\Rightarrow 512 - 2058 + 3240 - 20f(5) + 960 - 6f(3) + 8 = 0$

$\Rightarrow 20f(5) + 6f(3) = 2662$

$\Rightarrow 10f(5) + 3f(3) = 1331 \longrightarrow (iv)$

(iii)×10 $\Rightarrow 10f(5) + 10f(3) = 1520 \longrightarrow (v)$

Subtracting, $\quad -7f(3) = -189$

$\Rightarrow f(3) = 27$

Putting the value of $f(3)$ in (iii), we get

$f(5) + 27 = 152$

$\Rightarrow f(5) = 125$

Thus the missing terms are 27 and 125

Ex. Estimate the missing term in the following table:

x :	0	1	2	3	4
$y = f(x)$:	1	3	9	—	81

Sol^n (By 2nd method i.e. by the above method)

We are given 4 values, so $\Delta^4 f(x) = 0, \ \forall x$

$\Rightarrow (E-1)^4 f(x) = 0, \ \forall (x)$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0$

$\Rightarrow f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$

Putting $x = 0$, we get

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$\Rightarrow 81 - 4f(3) + 6\times2 - 4\times3 + 1 = 0$, by the given table.

$\Rightarrow 4f(3) = 124 \Rightarrow f(3) = 31.$