Linear Differential Equations

A differential eqn. of the form

$$\frac{dy}{dx} + Py = Q \longrightarrow (1)$$

where P and Q are functions of x or constants, is called the linear differential equation of the first order.

To solve (1), we multiply both sides by $e^{\int P dx}$. Then (1) becomes

$$e^{\int P dx} \frac{dy}{dx} + Py\, e^{\int P dx} = Q e^{\int P dx}.$$

or, $$\frac{d}{dx}\left[y e^{\int P dx} \right] = Q e^{\int P dx}$$

Integrating both sides w.r.t. x, we get

$$y e^{\int P dx} = \int [Q e^{\int P dx}] dx + C \rightarrow (2)$$

or, $$y\,(\text{I.F.}) = \int Q.\,(\text{I.F})\, dx + C.$$

where, $$\text{I.F. (Integrating Factor)} = e^{\int P dx}$$

(2) is the general soln. of (1).

<u>Integrating factor (I.F.)</u>: If a differential eqn. becomes integrable after multiplying a fn. of x or a constant (i.e. a factor), then such a factor is called an integrating factor of the diff. eqn.

Remark: Sometimes a differential eqn. takes linear form if we regard x as dependent variable and y as independent variable. In this case the eqn. is

$$\frac{dx}{dy} + Px = Q$$

where P and Q are functions of y or constants.

Here

$$I.F. = e^{\int P\,dy}$$

and the solutions is

$$x \cdot e^{\int P\,dy} = \int \left[Q \cdot e^{\int P\,dy}\right] dy + c.$$

Example 1. Solve

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

Soln. The given eqn. is

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{(1-x^2)}\,y = \frac{1}{(1-x^2)},$$

or, $$\frac{dy}{dx} + Py = Q$$

where, $$P = -\frac{x}{(1-x^2)}, \quad Q = \frac{1}{(1-x^2)}.$$

Now,

$$I.F. = e^{\int P\,dx}$$

$$= e^{\int \frac{-x}{(1-x^2)}\,dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$= \sqrt{1-x^2}$$

$\therefore$ General solution is

$$y.(I.F.) = \int Q.(I.F.)\,dx + c$$

or, $$y.(\sqrt{1-x^2}) = \int \frac{1}{1-x^2}.\sqrt{1-x^2}\,dx + c$$

or, $$y(\sqrt{1-x^2}) = \int \frac{1}{\sqrt{1-x^2}}\,dx + c$$

or, $$y(\sqrt{1-x^2}) = \sin^{-1}x + c$$

Which is the general solution.

Ex. 2. Solve $x\frac{dy}{dx} + 2y = x^2 \log x$

Soln: The given eqn. can be written as

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x$$

or, $$\frac{dy}{dx} + Py = Q, \quad \text{where } P = \frac{2}{x},\ Q = x\log x$$

which is linear.

Its $$I.F. = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$$

$\therefore$ G.S. is

$$y.(I.F.) = \int Q(I.F.)\,dx + c$$

or, $$y.x^2 = \int (x\log x).x^2\,dx + c$$

$$= \int x^3 \log x\,dx + c$$

$$= \log x.\frac{x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4}\,dx + c$$

(integration by parts.)

or, $$y.x^2 = \frac{1}{4}x^4\log x - \frac{1}{16}x^4 + c$$

or, $$y = \frac{1}{4}x^2\log x - \frac{1}{16}x^2 + cx^{-2} \leftarrow \text{Ans.}$$

Formula: $\int uv\,dx = u\int v\,dx - \int(u'\int v\,dx)\,dx + c$

Solve: $x\frac{dy}{dx}+2y=\frac{dy}{dx}+4$

Sol^n: given eqn is

$$x\frac{dy}{dx}+2y=\frac{dy}{dx}+4$$

or, $x\frac{dy}{dx}-\frac{dy}{dx}+2y=4$

or, $(x-1)\frac{dy}{dx}+2y=4$

or, $\frac{dy}{dx}+\frac{2}{x-1}y=\frac{4}{x-1}$

Which is linear.

Its. $I.F.=e^{\int\frac{2}{x-1}dx}=e^{2\log(x-1)}=(x-1)^2$

$\therefore$ G.S. is

$$y.(x-1)^2=\int\frac{4}{x-1}\times(x-1)^2dx+c$$

$$=\int 4(x-1)\,dx+c$$

or, $y(x-1)^2=2(x-1)^2+c$

Which is the required general sol^n. #

H.W. Solve: Ans:

1. $x\frac{dy}{dx}+2y=x^4$ $\quad[y.x^2=c+\frac{1}{6}x^6]$
2. $x(x-1)\frac{dy}{dx}-y=x^2(x-1)^2$ $\quad[y\frac{x}{x-1}=c+\frac{1}{3}x^3]$
3. $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$ $\quad[y=\tan^{-1}x-1+ce^{-\tan^{-1}x}]$
4. $\frac{dy}{dx}+y\tan x=2x+x^2\tan x$ $\quad[y\sec x=x^2\sec x+c]$