Prove that

5(1)

(1) $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$.

Proof: We have,

$$u_x - \Delta^n u_{x-n} = u_x - \Delta^n E^{-n} u_x$$ , interval of difference is one.

$$= u_x - \frac{\Delta^n}{E^n} u_x$$

$$= \left(1 - \frac{\Delta^n}{E^n}\right) u_x$$

$$= \frac{E^n - \Delta^n}{E^n} u_x$$

$$= \frac{1}{E^n}\left[\frac{E^n - \Delta^n}{1}\right] u_x$$

$$= \frac{1}{E^n}\left[\frac{E^n - \Delta^n}{E - \Delta}\right] u_x \quad , \because 1 + \Delta = E \Rightarrow 1 = E - \Delta.$$

$$= E^{-n}\left[E^{n-1} + E^{n-2}.\Delta + E^{n-3}\Delta^2 + \cdots + \Delta^{n-1}\right] u_x$$

$$= \left[E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \cdots + \Delta^{n-1} E^{-n}\right] u_x$$

$$= u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n}.$$

Hence,

$$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}.$$

(ii) $u_0 + {}^xC_1 \Delta u_1 + {}^xC_2 \Delta^2 u_2 + \cdots = u_x + {}^xC_1 \Delta^2 u_{x-1} + {}^xC_2 \Delta^4 u_{x-2} + \cdots$

$$\text{R.H.S.} = u_x + {}^xC_1 \Delta^2 u_{x-1} + {}^xC_2 \Delta^4 u_{x-2} + \cdots$$

$$= u_x + {}^xC_1 \Delta^2 E^{-1} u_x + {}^xC_2 \Delta^4 E^{-2} u_x + \cdots$$

$$= \left[1 + {}^xC_1 \Delta^2 E^{-1} + {}^xC_2 \Delta^4 E^{-2} + \cdots\right] u_x$$

$$= (1 + \Delta^2 E^{-1})^x u_x$$ , (By Binomial theorem).

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n + \cdots$

$$= \left(1 + \frac{\Delta^2}{E}\right)^x u_x$$

$$= \left(\frac{E + \Delta^2}{E}\right)^x u_x$$

$$= \left[\frac{E+(E-1)^2}{E}\right]^x u_x \quad , \because 1+\Delta = E \Rightarrow \Delta = E-1.$$

$$= \left[\frac{E+E^2-2E+1}{E}\right]^x u_x$$

$$= \left[\frac{E^2-E+1}{E}\right]^x u_x$$

$$= \frac{1}{E^x}\left[E(E-1)+1\right]^x u_x$$

$$= E^{-x}\left[E\Delta+1\right]^x u_x \quad , \because 1+\Delta = E \Rightarrow E-1 = \Delta$$

$$= E^{-x}\left[1+{}^xC_1 E\Delta + {}^xC_2 E^2\Delta^2 + \cdots\right] u_x$$

$$= \left[E^{-x} + {}^xC_1 \Delta E^{1-x} + {}^xC_2 \Delta^2 E^{2-x} + \cdots\right] u_x$$

$$= u_{x-x} + {}^xC_1 \Delta u_{x+1-x} + {}^xC_2 \Delta^2 u_{x+2-x} + \cdots$$

$$= u_0 + {}^xC_1 \Delta u_1 + {}^xC_2 \Delta^2 u_2 + \cdots$$

$$= \text{L.H.S.} \quad //$$

(iii) $u_0 + u_1 + u_2 + \cdots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \cdots + \Delta^n u_0$

L.H.S. $= u_0 + u_1 + u_2 + u_3 + \cdots + u_n$

$$= u_0 + Eu_0 + E^2 u_0 + \cdots + E^n u_0$$

$$= \left[1 + E + E^2 + \cdots + E^n\right] u_0$$

$$= \frac{1\left[E^{n+1}-1\right]}{E-1} u_0 \quad , \text{(sum of G.P.)}$$

Here,
$a = 1$
$r = E$
$n = n+1$
$S_n = a\cdot\frac{r^n-1}{r-1}$

$$= \frac{(1+\Delta)^{n+1}-1}{\Delta} u_0 \quad , \because 1+\Delta = E$$

$$= \frac{1}{\Delta}\left[1 + {}^{n+1}C_1 \Delta + {}^{n+1}C_2 \Delta^2 + \cdots + {}^{n+1}C_{n+1}\Delta^{n+1} - 1\right] u_0$$

$$= \left[{}^{n+1}C_1 + {}^{n+1}C_2 \Delta + {}^{n+1}C_3 \Delta^2 + \cdots + 1\cdot\Delta^n\right] u_0$$

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$$= {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \cdots + \Delta^n u_0.$$

$= R.H.S.$ //

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V.V.E. (iv) $u_0 - u_1 + u_2 - u_3 + \cdots = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 + \cdots$

Proof:

$$R.H.S. = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \cdots$$

$$= \frac{1}{2}\left[u_0 - \frac{1}{2}\Delta u_0 + \frac{1}{4}\Delta^2 u_0 - \frac{1}{8}\Delta^3 u_0 + \cdots\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\Delta + \frac{1}{4}\Delta^2 - \frac{1}{8}\Delta^3 + \cdots\right]u_0$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\Delta\right)^{-1} u_0, \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= \frac{1}{2}\left(\frac{2+\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2} \cdot \frac{(2+\Delta)^{-1}}{2^{-1}} u_0$$

$$= (2 + E - 1)^{-1} u_0, \quad \because 1 + \Delta = E \Rightarrow \Delta = E - 1.$$

$$= (1+E)^{-1} u_0$$

$$= (1 - E + E^2 - E^3 + \cdots) u_0$$

$$= u_0 - u_{0+1} + u_{0+2} - u_{0+3} + \cdots$$

$$= u_0 - u_1 + u_2 - u_3 + \cdots$$

$= L.H.S.$ //

V.V. Imp (v) $\Delta^n y_x = y_{x+n} - {}^nC_1 y_{x+n-1} + {}^nC_2 y_{x+n-2} \cdots + (-1)^n y_x$

$$R.H.S. = y_{x+n} - {}^nC_1 y_{x+n-1} + {}^nC_2 y_{x+n-1} + \cdots + (-1)^n y_x.$$

$$= E^n y_x - {}^nC_1 E^{n-1} y_x + {}^nC_2 E^{n-2} y_x + \cdots + (-1)^n y_x.$$

$$= [E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} + \cdots + (-1)^n] y_x$$

$$= (E-1)^n y_x = \Delta^n y_x = L.H.S.$$ //

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