Equations Reducible to Homogeneous Form :

An eqn. of the type

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}, \text{ when } \frac{a}{a'} \neq \frac{b}{b'} \longrightarrow (1)$$

can be reduced to homogeneous form as follows:

We put

$$\left.\begin{aligned} x &= X+h, \\ y &= Y+k. \\ \text{Then } \frac{dy}{dx} &= \frac{dY}{dX} \end{aligned}\right\} \longrightarrow (2)$$

where X, Y are new variables and h, k are arbitrary constants.

Substituting (2) in (1), we get:

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

We choose the constants h and k s.t.

$$ah+bk+c=0,$$
$$a'h+b'k+c'=0$$

With these substitution the differential reduces to

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$

which is homogeneous eqn. in X and Y and can be solved by putting $Y = vX$ as earlier.

Special case: When

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \quad (\text{say})$$

then the diff. eqn (1) can be written as

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c} \longrightarrow (3)$$

Then we put

$$ax+by = v$$

so that

$$a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

Then eqn. (3) becomes

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = \frac{v+c}{mv+c}$$

in which variables can be separated.

Ex. 1. Solve

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \longrightarrow (1) \quad \left(\text{here, } \frac{a}{a'} \neq \frac{b}{b'}\right)$$

Sol: We put

$$x = X+h$$

$$y = Y+k$$

where h and k are constants. Then

$$\frac{dy}{dx} = \frac{dY}{dX}.$$

$\therefore$ The given eqn. becomes

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}$$

$$= \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \longrightarrow (2)$$

Now, we choose h and k such that

$$h + 2k - 3 = 0, \longrightarrow (i)$$

and $$2h + k - 3 = 0 \longrightarrow (ii)$$

Now, $(i) \times 2 \Rightarrow$

$$2h + 4k - 6 = 0 \longrightarrow (iii)$$

$(iii) - (ii) \Rightarrow 3k - 3 = 0$

$$\Rightarrow k = 1$$

putting $k = 1$ in (i),

$$h + 2 \cdot 1 - 3 = 0$$

$$\Rightarrow h = 1.$$

$\therefore$ From (2),

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \longrightarrow (3)$$

which is homogeneous in X and Y.

Now, we put

$$Y = vX$$

$$\therefore \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$\therefore (3) \Rightarrow$ $$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + vX} = \frac{1 + 2v}{2 + v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 + 2v - 2v - v^2}{2 + v} = \frac{1 - v^2}{2 + v}$$

$$\Rightarrow \frac{dX}{X} = \frac{2 + v}{1 - v^2}\,dv$$, (variables are separated)

$$= \left(\frac{2}{1 - v^2} + \frac{v}{1 - v^2}\right)dv$$

Integrating, we get

$$\log X = 2 \cdot \frac{1}{2}\log\frac{1 + v}{1 - v} - \frac{1}{2}\log(1 - v^2) + \log c$$

$$\left[\because \int\frac{dx}{1 - x^2} = \frac{1}{2}\log\frac{1 + x}{1 - x}\right]$$

or, $$X = c\,\frac{1+v}{1-v}\cdot\frac{1}{\sqrt{1-v^2}}$$

$$= \frac{c\sqrt{1+v}}{(1-v)^{3/2}}$$

or, $$X^2 = \frac{c^2(1+v)}{(1-v)^3}$$

or, $$X^2(1-v)^3 = c^2(1+v)$$

or, $$X^2\left(1-\frac{Y}{X}\right)^3 = c^2\left(1+\frac{Y}{X}\right), \quad \because v = \frac{Y}{X}$$

or, $$X^2\left(\frac{X-Y}{X}\right)^3 = c^2\left(\frac{X+Y}{X}\right)$$

or, $$(X-Y)^3 = c^2(X+Y)$$

or, $$(x-1-y+1)^3 = c^2(x-1+y-1), \quad \because x = X+1,\ y = Y+1$$

or, $$(x-y)^3 = c^2(x+y-2)$$

which is the required solution.

H.W. Solve:

① $(3x-7y-3)\dfrac{dy}{dx} = 3y-7x+7$

[Ans: $(y-x+1)^2(y+x-1)^5 = c$]

② $(2x+y+3)\dfrac{dy}{dx} = x+2y+3$

[Ans: $(x-y)^3 = c(x+y-1)$]

(3) $(x+y)(dx-dy) = dx+dy$

[Ans: $\log(x+y) = x-y+c$]

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