Ex. Prove that

$$\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$

using the result $\Delta - \nabla = \Delta\nabla$.

Solⁿ.

$$\text{R.H.S.} = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$

$$= \frac{\Delta^2 - \nabla^2}{\nabla\Delta}$$

$$= \frac{(\Delta+\nabla)(\Delta-\nabla)}{\Delta-\nabla}, \quad \because \Delta - \nabla = \Delta\nabla, \text{ given}$$

$$= \Delta + \nabla$$

$$= \text{L.H.S.}$$

Proved.

Ex. Prove that

$$\left(\frac{\Delta^2}{E}\right)x^3 = 6x, \quad 1 \text{ being the interval}$$

of differences.

Solⁿ.

$$\text{L.H.S.} = \left(\frac{\Delta^2}{E}\right)x^3$$

$$= \left\{\frac{(E-1)^2}{E}\right\}x^3$$

$$= \left(\frac{E^2-2E+1}{E}\right)x^3$$

$$= (E - 2 + E^{-1})x^3$$

$$= E(x^3) - 2x^3 + E^{-1}(x^3)$$

$$= (x+1)^3 - 2x^3 + (x-1)^3$$

$$= \cancel{x^3} + \cancel{3x^2} + 3x + \cancel{1} - \cancel{2x^3} + \cancel{x^3} - \cancel{3x^2} + 3x - \cancel{1}$$

$$= 6x = \text{R.H.S.}$$

Ex. Evaluate $\frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x}$, h being the interval of differences.

Sol^n.

$$\frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x}$$

$$= (\Delta^2 E^{-1}) e^x \frac{Ee^x}{\Delta^2 e^x}$$

$$= \Delta^2 e^{x-h} \frac{e^{x+h}}{\Delta^2 e^x}$$

$$= \Delta^2 e^x . e^{-h} \frac{e^{x+h}}{\Delta^2 e^x}$$

$$= e^{-h} \Delta^2 e^x \frac{e^{x+h}}{\Delta^2 e^x}$$

$$= e^{-h} . e^{x+h}$$

$$= e^x \longleftarrow \text{Ans.}$$

Ex. Show that

$$\Delta \log f(x) = \log \left\{ 1 + \frac{\Delta f(x)}{f(x)} \right\}$$

Sol^n.

$$\text{L.H.S.} = \Delta \log f(x)$$

$$= \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)} = \log . \frac{E f(x)}{f(x)}$$

$$= \log \left\{ \frac{(1+\Delta) f(x)}{f(x)} \right\}, \quad \because E \equiv 1+\Delta$$

$$= \log \left\{ 1 + \frac{\Delta f(x)}{f(x)} \right\}$$

$$= \text{R.H.S.}$$

Ex. Evaluate $\Delta^2(3e^x)$

Sol^n. $\Delta(3e^x) = 3(\Delta e^x)$

$= 3(e^{x+h} - e^x)$

$= 3e^x(e^h - 1)$

$\therefore \Delta^2(3e^x) = \Delta\{\Delta(3e^x)\}$

$= \Delta\{3e^x(e^h-1)\}$

$= 3(e^h-1)\,\Delta e^x$

$= 3(e^h-1)(e^{x+h} - e^x)$

$= 3(e^h-1)(e^h-1)e^x$

$= 3(e^h-1)^2 e^x \leftarrow$ Ans.

H.W. Evaluate

(1) $\Delta^2(\cos 2x)$. ; Ans: $-4\sin^2 h \cos(2x+2h)$

(2) $\Delta(x+\cos x)$; Ans: $h - 2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

(3) $\Delta \tan^{-1} x$; Ans: $\tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$

(4) Prove that $\left(\frac{\Delta^2}{E}\right)x^3 = 6h^2x$

where h is the interval of differences.

Ex. Show by induction that

$$\Delta^n \sin(a+bx) = \left(2\sin\frac{b}{2}\right)^n \sin\left\{a+bx+\frac{n}{2}(b+\pi)\right\}$$

Solⁿ: We have

$$\Delta \sin(a+bx) = \sin\{a+b(x+1)\} - \sin(a+bx)$$, where interval of diff. is 1

$$= 2\cos\frac{2a+2bx+b}{2}\sin\frac{b}{2}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \left(2\sin\frac{b}{2}\right)\sin\left\{\frac{\pi}{2}+\left(a+bx+\frac{b}{2}\right)\right\}$$

$$= 2\sin\frac{b}{2}\sin\left[a+bx+\frac{1}{2}(b+\pi)\right]$$

This show that the result is true for $n=1$.

Let us assume that the result is true for $n=k$, ($k \in N$). i.e.

$$\Delta^k \sin(a+bx) = \left(2\sin\frac{b}{2}\right)^k \sin\left\{a+bx+\frac{k}{2}(b+\pi)\right\}$$

Now, operating by Δ on both sides, we get

$$\Delta^{k+1}\sin(a+bx) = \Delta\left(2\sin\frac{b}{2}\right)^k \sin\left\{a+bx+\frac{k}{2}(b+\pi)\right\}$$

$$= \left(2\sin\frac{b}{k}\right)^k\left[\sin\left\{a+b(x+1)+\frac{k}{2}(b+\pi)\right\} - \sin\left\{a+bx+\frac{k}{2}(b+\pi)\right\}\right]$$

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or,

$$\Delta^{k+1}\sin(a+bx) = \left(2\sin\frac{b}{2}\right)^{k}\left[2\cos\frac{2\{a+bx+\frac{k}{2}(b+\pi)\}+b}{2}\sin\frac{b}{2}\right]$$

$$= \left(2\sin\frac{b}{2}\right)^{k+1}\cos\left\{a+bx+\frac{k}{2}(b+\pi)+\frac{b}{2}\right\}$$

$$= \left(2\sin\frac{b}{2}\right)^{k+1}\sin\left[\frac{\pi}{2}+\left\{a+bx+\frac{k}{2}(b+\pi)+\frac{b}{2}\right\}\right]$$

$$= \left(2\sin\frac{b}{2}\right)^{k+1}\sin\left[a+bx+\frac{1}{2}(\pi+kb+k\pi+b)\right]$$

$$= \left(2\sin\frac{b}{2}\right)^{k+1}\sin\left[a+bx+\frac{k+1}{2}(b+\pi)\right]$$

$\therefore$ The result is true for $k+1$.

Hence by mathematical induction, the result is true $\forall\, n \in \mathbb{N}$. # Proved.

H.W. Show by mathematical induction that

$$\Delta^{n}\cos(a+bx) = \left(2\sin\frac{b}{2}\right)^{n}\cos\left\{a+bx+\frac{n}{2}(b+\pi)\right\}.$$