Homogeneous differential Equations

An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$

or, $f_1(x,y)\,dx + f_2(x,y)\,dy = 0$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree can be reduced to an equation in which variables are separable by putting

$$\boxed{y = vx} \longrightarrow (1)$$

so that, on differentiating (1), we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \longrightarrow (2)$$

Putting these values of y and $\frac{dy}{dx}$ in the given eqn., it reduces to eqn which is variables are separable & can be solved.

<u>Note</u>: <u>Homogenous fn</u>: A function $f(x,y)$ is said to be homogeneous of n^{th} degree if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)\,, \quad \text{where } \lambda \text{ is const.}$$

<u>e.g. ①</u> Let $f(x,y) = x^2 + y^2 + 2xy$

Now, $f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 + 2\lambda x \cdot \lambda y$

$= \lambda^2(x^2 + y^2 + 2xy)$

$= \lambda^2 f(x,y)$

$\therefore f(x,y)$ is a homo fn. of 2^{nd} degree.

Ex1. Solve

$$(x^2+y^2)\,dx + 2xy\,dy = 0$$

Soln. Given eqn is

$$(x^2+y^2)\,dx + 2xy\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2+y^2}{2xy} \rightarrow (1)$$

Here $f_1(x,y) = x^2+y^2$ which is homo. fn of degree 2. Also, $f_2(x,y) = 2xy$ which is also homo. fn. of degree 2.

which is a homo eqn.

So we put

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}, \text{ diff. w.r.t. } x$$

Now, putting the values of y and $\frac{dy}{dx}$ in the given eqn (1), we get

$$v + x\frac{dv}{dx} = -\frac{x^2+x^2v^2}{2\cdot x\cdot vx}$$

$$= -\frac{x^2(1+v^2)}{x^2\cdot 2v}$$

$$= -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -v - \frac{1+v^2}{2v}$$

$$= \frac{-2v^2-1-v^2}{2v} = \frac{-1-3v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \frac{dx}{x} = -\frac{2v}{1+3v^2}\,dv$$ (variables are separated now)

Integrating, $\log x + \frac{1}{3}\log(1+3v^2) = \log C$

$$\Rightarrow x(1+3v^2)^{1/3} = C \quad \Rightarrow x(1+3\,y^2/x^2)^{1/3} = C$$

which is the G. Soln. #

Ex. 2. Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol^n: From given eqn,

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2y}{x^3+y^3} \rightarrow (1) \text{, which homogeneous.}$$

$\therefore$ Putting $y = vx$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ Given eqn. (1) becomes

$$v + x\frac{dv}{dx} = -\frac{x^2 \cdot vx}{x^3+v^3x^3} = -\frac{x^3v}{x^3(1+v^3)}$$

$$= -\frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = -v - \frac{v}{1+v^3} = -\frac{v+v^4-v}{1+v^3} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \frac{dx}{x} = -\frac{1+v^3}{v^4}dv = -\left[\frac{1}{v^4} + \frac{1}{v}\right]dv$$

(variables are separated)

Integrating, we get

$$\log x = \frac{1}{3v^3} - \log v + c$$

$$\Rightarrow \log vx = \frac{1}{3v^3} + c$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} + c, \quad \because y = vx$$

which is the general sol^n

H.W. Solve:

(i) $\dfrac{dy}{dx} = \dfrac{y^3 + 3x^2y}{x^3 + 3xy^2}$

(ii) $y^2 + x^2 \dfrac{dy}{dx} = xy \dfrac{dy}{dx}$

(iii) $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0$

(iv) $\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \dfrac{x}{y}\right) dy = 0$

(v) $y^2\,dx + (x^2 + xy + y^2)\,dy = 0$

(vi) $(x^2 + y^2)\,dy = xy\,dx.$