Method of sol^n. of 1st order 1st degree diff. Eqns:

Found in H.S. Course:
- (i) Separation of variables
- ii) homogeneous diff. eqns.
- iii) Linear differential eqns. / Bernoulli's Eqn. [eqns. reducible to Lin form]
- (iv) Exact Differential Eqn.

Method-I : Separation of variables

[First we separate the variables, then integrate]

Example 1. Solve

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

Soln. Given eqn is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

or, $$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{[variables are separated]}$$

Integrating, we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + c$$

or, $$\tan^{-1} y = \tan^{-1} x + C$$

or, $$\tan^{-1} y - \tan^{-1} x = C$$

or, $$\tan^{-1} \frac{y-x}{1+xy} = C = \tan^{-1} A \quad (\text{say})$$

or, $$\frac{y-x}{1+xy} = A \Rightarrow y = x + A(1+xy) \leftarrow \text{Ans}$$

which is the general sol^n.

Ex. 2. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solⁿ. Given eqn. is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$= \frac{e^x}{e^y} + \frac{x^2}{e^{+y}}$$

$$\Rightarrow e^y dy = (e^x + x^2)\, dx$$

(variables are separated)

Integrating,

$$\int e^y dy = \int (e^x + x^2)\, dx + c$$

or, $$e^y = e^x + \frac{1}{3} x^3 + c$$

Which is the general solⁿ.

H.W.

Solve the following diff. eqns:

1) $(x^2 - y x^2) dy + (y^2 + x y^2) dx = 0$

2) $(1+y^2)(1+\log x)\, dx + x\, dy = 0$

3) $(1+e^{2x}) dy + (1+y^2) e^x dx = 0$

4) $\frac{dy}{dx} = xy + x + y + 1$

5) $\frac{dy}{dx} = y \sin 2x$

3. Equations Reducible to the form is which Variables are separable:

Eqns. of the form

$$\frac{dy}{dx} = f(ax+by+c) \longrightarrow (1)$$

In this case, we put

$$ax+by+c = v$$

so that

$$a + b\frac{dy}{dx} = \frac{dv}{dx}$$

i.e. $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

Then the eqn (1) becomes

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$$

or, $$\frac{dv}{dx} = a + b f(v)$$

in which variables are separable.

Example: Solve

$$\frac{dy}{dx} = (4x+y+1)^2$$

Solⁿ: Let, $4x+y+1 = v$

so that $4 + \frac{dy}{dx} = \frac{dv}{dx}$ (diff. w.r.t. x)

∴ Given eqn. reduces to

$$\frac{dv}{dx} - 4 = v^2$$

or, $\frac{dv}{dx} = v^2+4$

or, $\frac{dv}{v^2+4} = dx$, variables are separated.

Integrating,

$$\int \frac{dv}{v^2+4} = \int dx + c$$

or, $\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + c$

or, $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$

which is the required general solⁿ.

Ex. 2. Solve

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Solⁿ. Given eqⁿ is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Now we put

$$x+y = v$$

so that $1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ Given eqn becomes

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

or, $$\frac{dv}{dx} = 1 + \sin v + \cos v$$

or, $$dx = \frac{dv}{1 + \sin v + \cos v}$$ [variables are separated]

$$= \frac{dv}{(1+\cos v) + \sin v}$$

$$= \frac{dv}{2\cos^2 \frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$1 + \cos 2\theta = 2\cos^2\theta$

$$= \frac{dv}{2\cos^2\frac{v}{2}\left(1 + \tan\frac{v}{2}\right)}$$

or, $$dx = \frac{\frac{1}{2}\sec^2\frac{v}{2}\, dv}{1 + \tan\frac{v}{2}}$$

$\int \tan x\, dx = \sec^2 x + c$

Integrating, we get

$$x + c = \log\left(1 + \tan\frac{v}{2}\right) \quad \because$$

$\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$

or, $$x + c = \log\left[1 + \tan\tfrac{1}{2}(x+y)\right]$$

$, [\because x + y = v$

which is the required general solution.

Ex. 3. Solve

$$\frac{dy}{dx} = \sin(x+y)$$

Sol^n: Let

$$v = x+y$$

$$\therefore \frac{dv}{dx} = 1 + \frac{dy}{dx}.$$

$\therefore$ Given eqn. reduces to (on putting this value)

$$\frac{dv}{dx} - 1 = \sin v \Rightarrow \frac{dv}{dx} = 1 + \sin v$$

On separating variables, we get

$$dx = \frac{dv}{1+\sin v}$$

$$\Rightarrow \int dx + c = \int \frac{dv}{1+\sin v}$$

$$\Rightarrow x + c = \int \frac{(1-\sin v)}{(1+\sin v)(1-\sin v)} dv$$

$$= \int \frac{1-\sin v}{\cos^2 v} dv$$

$$= \int \frac{1}{\cos^2 v} dv - \int \frac{\sin v}{\cos^2 v} dv$$

$$= \int \sec^2 v \, dv - \int \sec v \tan v \, dv$$

$$= \tan v - \sec v$$

$$\Rightarrow x + c = \tan(x+y) - \sec(x+y),$$ which is h.s.

H.W. Solve

(1) $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$

(2) $\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$

(3) $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$

(4) $\frac{dy}{dx} = (x+y)^2$