Backward or Ascending Differences :-

The differences

$$f(a+h) - f(a), \quad f(a+2h) - f(a+h), \ldots$$

$$f(a+nh) - f(a+\overline{n-1}\,h)$$

are called first backward differences and these are denoted by $\nabla f(a+h)$, $\nabla f(a+2h), \ldots, \nabla f(a+nh)$ respectively so that

$$\nabla f(a+h) = f(a+h) - f(a)$$

$$\nabla f(a+2h) = f(a+2h) - f(a+h) \text{ etc.}$$

where ∇ (nebla) is called backward difference operator.

The differences of first backward differences are said to be the second backward differences and are denoted by $\nabla^2 f(a+2h)$, $\nabla^2 f(a+3h)$ etc.

i.e. $\nabla^2 f(a+2h) = \nabla f(a+2h) - \nabla f(a+h)$

$$= [f(a+2h) - f(a+h)] - [f(a+h) - f(a)]$$

$$= f(a+2h) - 2f(a+h) + f(a)$$

Scanned with CamScanner

In general, first backward difference is given by

$$\nabla f(x) = f(x) - f(x-h).$$

Again, the second backward difference is given by

$$\nabla^2 f(x) = \nabla f(x) - \nabla f(x-h)$$

$$= [f(x) - f(x-h)] - [f(x-h) - f(x-2h)]$$

$$= f(x) - 2f(x-h) + f(x-2h).$$

In the same way it can be shown that the n^{th} backward difference is given by

$$\nabla^n f(x) = \nabla [\nabla^{n-1} f(x)]$$

$$= \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h).$$

Shift operator (Displacement operator)

The shift operator is denoted by E and defined by

$$E f(x) = f(x+h).$$

In particular,

$$E f(a) = f(a+h)$$

$$E f(a+h) = f(a+2h) \text{ etc.}$$

Scanned with CamScanner

Fundamental Theorem of Difference Calculus :-

The n^{th} order difference of a polynomial of degree n is constant and higher order differences are zero.

Proof :-

Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \,, \quad a_0 \neq 0$$

be a polynomial of degree n.

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= a_0 (x+h)^n + a_1 (x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n$$

$$- \left[a_0 x^n + a_1 x^n + \cdots + a_{n-1} x + a_n \right]$$

$$= a_0 \left[(x+h)^n - x^n \right] + a_1 \left[(x+h)^{n-1} - x^{n-1} \right]$$

$$+ \cdots + a_{n-1} \left[x + h - x \right]$$

$$= a_0 \left[{}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \cdots \right]$$

$$+ a_1 \left[{}^{n-1}C_1 x^{n-2} h + {}^{n-1}C_2 x^{n-3} h^2 + \cdots \right]$$

$$+ \cdots + a_{n-1} h$$

$$= a_0 \, {}^nC_1 h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-1} x + b_n$$

where $b_2, b_3, \ldots, b_n$ are consts. and $b_n = a_{n-1} h$.

Scanned with CamScanner

or,

$$\Delta f(x) = a_0 nh x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_{n-1} x + b_n$$

which is a polynomial of degree $n-1$.

Again,

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$= a_0 nh[(x+h)^{n-1} - x^{n-1}]$$

$$+ b_2 [(x+h)^{n-2} - x^{n-2}] + \cdots$$

$$+ b_{n-1} [(x+h) - x]$$

$$= a_0 nh [{}^{n-1}C_1 x^{n-2} h + {}^{n-1}C_2 x^{n-3} h^2 + \cdots]$$

$$+ b_2 [{}^{n-2}C_1 x^{n-3} h + {}^{n-2}C_2 x^{n-4} h^2 + \cdots]$$

$$+ \cdots + b_{n-1} h$$

$$= a_0 n(n-1) h^2 x^{n-2} + c_3 x^{n-3} + c_4 x^{n-4} + \cdots$$

$$+ c_{n-1} x + c_n ,$$

where $c_3, c_4, \ldots, c_n$ are constants.

Thus, we see that $\Delta^2 f(x)$ is a polynomial of degree $n-2$.

Proceeding in the same way we shall see that $\Delta^n f(x)$ is a poly. of degree $n-n$ i.e. zero, i.e. a constant. Lastly,

$$\Delta^n f(x) = a_0 n(n-1)(n-2)\ldots(n-\overline{n-1}) h^n x^0$$

$$= a_0 \underline{|n} h^n,$$ where h is the interval of difference.

Also, next difference

$$\Delta^{n+1} f(x) = 0 .$$

Proved.