Finite Difference Operators

Let $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$ be the $n+1$ values of the function $y = f(x)$ corresponding to the values of $a, a+h, a+2h, \ldots, a+nh$ of x. The values of x are called arguments and the values of y are called entries.

Forward or Descending Difference :

The differences $f(a+h) - f(a)$, $f(a+2h) - f(a+h)$, $\ldots$, $f(a+nh) - f(a+\overline{n-1}\,h)$ are called first order forward difference and are denoted by $\Delta f(a), \Delta f(a+h), \ldots, \Delta f(a+\overline{n-1}\,h)$ respectively, where Δ (delta) is called the forward or descending difference operator.

Thus we get,

$$\left.\begin{aligned} \Delta f(a) &= f(a+h) - f(a) \\ \Delta f(a+h) &= f(a+2h) - f(a+h) \\ &\ldots\ldots\ldots \\ \Delta f(a+\overline{n-1}\,h) &= f(a+nh) - f(a+\overline{n-1}\,h) \end{aligned}\right\} \rightarrow (1)$$

In general, the first order difference is given by

$$\Delta f(x) = f(x+h) - f(x)$$

The differences of first forward differences given by (1) are called the second forward differences and denoted by $\Delta^2 f(a)$, $\Delta^2 f(a+h)$, ----, $\Delta^2 f(a+\overline{n-1}h)$.

Thus, we have

$$\Delta^2 f(a) = \Delta f(a+h) - \Delta f(a)$$

$$= [f(a+2h)-f(a+h)] - [f(a+h) - f(a)]$$

$$= f(a+2h) - 2f(a+h) + f(a).$$

Similarly,

$$\Delta^2 f(a+h) = \Delta f(a+2h) - \Delta f(a+h)$$

$$= [f(a+3h) - f(a+2h)] - [f(a+2h) - f(a+h)]$$

$$= f(a+3h) - 2f(a+2h) + f(a+h)$$

Again, differences of the second forward differences are called <u>third forward differences</u> and are denoted by $\Delta^3 f(a)$, $\Delta^3 f(a+h)$, etc.

Thus,

$$\Delta^3 f(a) = \Delta^2 f(a+h) - \Delta^2 f(a)$$

$$= \Delta[f(a+2h) - \Delta f(a+h)]$$

$$- [\Delta f(a+h) - \Delta f(a)]$$

or,

$$\Delta^3 f(a) = [f(a+3h) - f(a+2h)] - [f(a+2h) - f(a+h)] - [f(a+2h) - f(a+h)] + [f(a+h) - f(a)]$$

$$= f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$$

<u>Note:</u> The co-efficients in the expressions of the differences $\Delta f(a)$, $\Delta^2 f(a)$, $\Delta^3 f(a)$, ..., are same as the co-efficients in the expressions of $(a-b)^1$, $(a-b)^2$, $(a-b)^3$, ... etc.

<u>In general</u>, the n^{th} forward difference is given by

$$\boxed{\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)}$$